4

ISING MODEL ON THE BETHE LATTICE

4.1 The Bethe Lattice

Another simple model that can be exactly solved is the Ising model (or indeed any model with only nearest-neighbour interactions) on the Bethe lattice. Like the mean-field model, this is equivalent to an approximate treatment of a model on, say, a square or cubic lattice (Bethe, 1935). However, it can be defined as an exactly solvable model, and this is what we shall do here.

Consider the graph constructed as follows: start from a central point 0 and add q points all connected to 0. Call the set of these q points the 'first shell'. Now create further shells by taking a point in shell r and connecting $q - 1$ new points to it. Do this for all points in shell r and call the set of all the new points 'shell $r + 1$'.

Proceeding interatively in this way, construct shells $2, 3, \ldots, n$. This gives a graph like that shown in Fig. 4.1. There are $q(q-1)^{r-1}$ points in shell r and the total number of points in the graph is

$$q[(q-1)^n - 1]/(q-2) \tag{4.1.1}$$

We call the points in shell n 'boundary points'. They are exceptional in that each has only one neighbour, while all other points (interior points) each have q neighbours.

Such a graph contains no circuits and is known as a Cayley tree. From our point of view it can be thought of as a regular 'lattice' of coordination number q (i.e. q neighbours per site), *provided the boundary sites can be ignored.*

There is a problem here: normally the ratio of the number of boundary sites to the number of interior sites of a lattice becomes small in the thermodynamic limit of a large system. Here it does not, since both numbers grow exponentially like $(q-1)^n$. To overcome this problem we here consider only local properties of sites *deep within the graph* (i.e. infinitely far from the boundary in the limit $n \to \infty$). Such sites should all be equivalent, each having coordination number q, and can be regarded as forming the *Bethe lattice*. (This distinction between the Cayley tree and the Bethe lattice is not always made, but does seem to be useful terminology. I am grateful to Professor J. Nagle for suggesting it to me and drawing my attention to a relevant article [Chen *et al.*, 1974].)

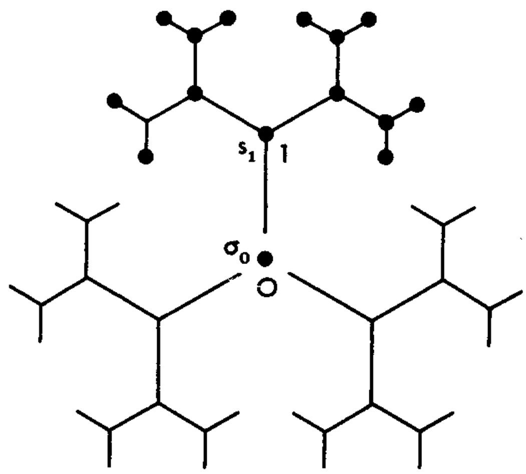

Fig. 4.1. A Cayley tree (with $q = 3$ and $n = 4$), divided at the central site 0 into three sub-trees. They are identical, but here the upper sub-tree is distinguished by indicating its sites with solid circles. Each sub-tree is rooted at 0. The site 1 adjacent to 0 in the upper sub-tree is shown. The spin at 0 is σ_0, that at 1 is s_1.

Put another way, if we construct an Ising model on the complete Cayley tree, then the partition function Z contains contributions from both sites deep within the graph, and sites close to or on the boundary. The contribution from the latter is not negligible, even in the thermodynamic limit.

If one considers the total partition function, then one is considering the 'Ising model on the Cayley tree'. This problem has been solved (Runnels, 1967; Eggarter, 1974; Müller-Hartmann and Zittartz, 1974) and has some quite unusual properties. We shall not, however, consider this problem here. Instead we shall effectively consider only the contribution to Z from sites deep within the graph, i.e. from the Bethe lattice.

Some motivation for this choice is given by series expansions. If one makes a low temperature expansion as in Section 1.8 for any regular lattice, then to second order the only properties of the lattice that one needs to know are the number of sites and the coordination number. To third order one needs the number of triangles in the lattice, to fourth order the number of tetrahedra (i.e. clusters of 4 sites all connected to one another) and other highly connected 4-point sub-graphs, and so on. An interesting simple case is when there are no circuits at all, and hence no triangles, tetrahedra, etc. Then one obtains the Ising model on the Bethe lattice as defined here.

4.2 Dimensionality

Consider any regular lattice. Let $m_1(=q)$ be the number of neighbours per site, m_2 the number of next-nearest neighbours, m_3 the number of next-next-nearest neighbours, etc. Then $c_n = 1 + m_1 + m_2 + \ldots + m_n$ is the number of sites within n steps of a given site. For the hyper-cubic lattices it is easy to see that

$$\lim_{n\to\infty} (\ln c_n)/\ln n = d\,, \tag{4.2.1}$$

where d is the dimensionality of the lattice.

The relation (4.2.1) is also true for all the regular two and three-dimensional lattices, and can be regarded as a definition of the dimensionality d.

Now return to considering the Bethe lattice. In this case c_n is given by (4.1.1). Substituting this expression into (4.2.1) gives $d = \infty$, so in this sense the Bethe lattice is 'infinite-dimensional'.

4.3 Recurrence Relations for the Central Magnetization

Consider an Ising model on the complete Cayley tree (but we shall later ignore boundary terms, thereby reducing it to the Bethe lattice). The partition function is given by (1.8.2), i.e. by

$$Z = \sum_{\sigma} P(\sigma)\,, \tag{4.3.1}$$

where

$$P(\sigma) = \exp\left[K \sum_{(i,j)} \sigma_i\sigma_j + h \sum_i \sigma_i\right]. \tag{4.3.2}$$

The first summation in (4.3.2) is over all edges of the graph, the second over all sites. The $P(\sigma)$ can be thought of as an unnormalized probability distribution: in particular, if σ_0 is the spin at the central site 0, then the local magnetization there is

$$M = \langle \sigma_0 \rangle = \sum_{\sigma} \sigma_0 P(\sigma)/Z\,. \tag{4.3.3}$$

From Fig. 4.1 it is apparent that if the graph is cut at 0, then it splits up into q identical disconnected pieces. Each of these is a rooted tree (with root 0). This implies that the expression (4.3.2) factors:

$$P(\sigma) = \exp(h\sigma_0) \prod_{j=1}^{q} Q_n(\sigma_0|s^{(j)})\,, \tag{4.3.4}$$

where $s^{(j)}$ denotes all the spins (other than σ_0) on the jth sub-tree, and

$$Q_n(\sigma_0|s) = \exp\left[K \sum_{(i,j)} s_i s_j + K s_1 \sigma_0 + h \sum_i s_i\right], \tag{4.3.5}$$

s_i being the spin on site i of the sub-tree (other than the root, which has spin σ_0). Site 1 is the site adjacent to 0, as in the upper sub-tree of Fig. 4.1. The first summation in (4.3.5) is over all edges of the sub-tree other than (0,1); the second is over all sites other than 0. The suffix n denotes the fact that the sub-tree has n shells, i.e. n steps from the root to the boundary sites.

Further if the upper sub-tree in Fig. 4.1 is cut at the site 1 adjacent to 0, then it too decomposes into q pieces: one being the 'trunk' (0, 1), the rest being identical branches. Each of these branches is a sub-tree like the original, but with only $n - 1$ shells. Thus

$$Q_n(\sigma_0|s) = \exp(K\sigma_0 s_1 + h s_1) \prod_{j=1}^{q-1} Q_{n-1}(s_1|t^{(j)}) \tag{4.3.6}$$

where $t^{(j)}$ denotes all the spins (other than s_1) on the jth branch of the sub-tree.

These factorization relations (4.3.4) and (4.3.6) make it easy to calculate M. Let

$$g_n(\sigma_0) = \sum_{s} Q_n(\sigma_0|s)\,. \tag{4.3.7}$$

Then from (4.3.1) and (4.3.4),

$$Z = \sum_{\sigma_0} \exp(h\sigma_0)\,[g_n(\sigma_0)]^q\,. \tag{4.3.8}$$

Similarly, from (4.3.3) and (4.3.4),

$$M = Z^{-1} \sum_{\sigma_0} \sigma_0 \exp(h\sigma_0)\,[g_n(\sigma_0)]^q\,. \tag{4.3.9}$$

Let

$$x_n = g_n(-)/g_n(+) . \tag{4.3.10}$$

Then from (4.3.8) and (4.3.9),

$$M = \frac{e^h - e^{-h} x_n^q}{e^h + e^{-h} x_n^q} . \tag{4.3.11}$$

Thus M is known if x_n is. To obtain x_n we sum (4.3.6) over all the spins s, i.e. over s_1 and the $t^{(j)}$, to give, using only (4.3.7):

$$g_n(\sigma_0) = \sum_{s_1} \exp(K\sigma_0 s_1 + hs_1) [g_{n-1}(s_1)]^{q-1} \tag{4.3.12}$$

Remembering that σ_0 and s_1 are single spins, with values $+1$ and -1, performing the summation in (4.3.12) for $\sigma_0 = +1$ or -1, taking ratios and using (4.3.10), we obtain

$$x_n = y(x_{n-1}) , \tag{4.3.13}$$

where the function $y(x)$ is given by

$$y(x) = [e^{-K+h} + e^{K-h} x^{q-1}]/[e^{K+h} + e^{-K-h} x^{q-1}] . \tag{4.3.14}$$

Equation (4.3.13) is a recurrence relation between x_n and x_{n-1}. It is easy to see that

$$x_0 = g_0(\sigma_0) = 1 , \tag{4.3.15}$$

so (4.3.13) defines x_n, and (4.3.11) defines M.

4.4 The Limit $n \to \infty$

Hereafter we consider the ferromagnetic case, $K > 0$. Then $y(x)$ increases monotonically from $\exp(-2K)$ to $\exp(2K)$ as x goes from 0 to ∞.

The recurrence relation (4.3.13) can be thought of graphically by simultaneously plotting $y = y(x)$ and $y = x$.

Let P_{n-1} be the point $(x_{n-1}, y(x_{n-1}))$ in the (x, y) plane. To construct P_n draw a horizontal line through P_{n-1} to intercept the line $y = x$ at a point Q_n. Now draw a vertical line through Q_n. Its intercept with $y = y(x)$ is the point P_n.

There are two cases to consider: either the line $y = x$ crosses the curve $y = y(x)$ once, or it crosses it three times, as shown in Fig. 4.2. In the former case the point P_n will always monotonically approach the cross-over point A as $n \to \infty$, as indicated in Fig. 4.2(a). Thus x_n and M tend to a limit as n becomes large, as we expect. This M is therefore the local magnetization of a site deep within the Cayley tree, i.e. the magnetization per site of the Bethe lattice.

If there are three cross-over points, then the outer two (A and C in Fig. 4.2(b)) are stable limit points of (4.3.13), while the centre one (B) is unstable. If P_0 lies to the left (right) of B, then P_n tends to A (C). Thus again P_n tends to a limit, giving the magnetization M for the Bethe lattice.

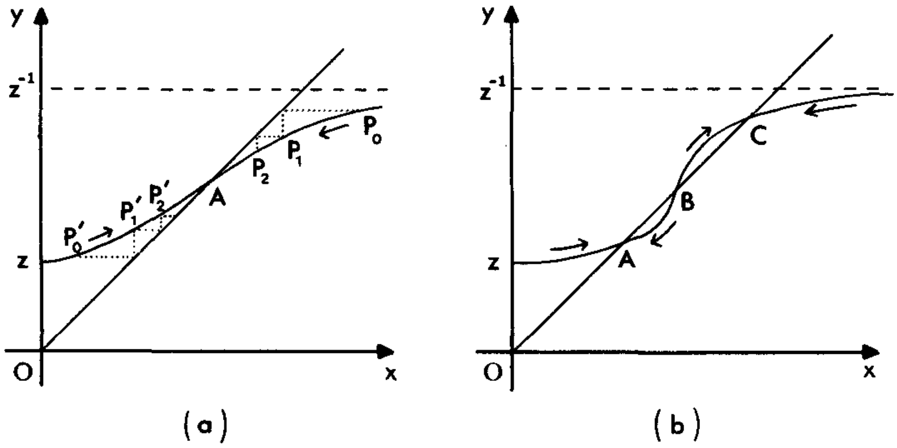

Fig. 4.2. Typical sketches of the function $y(x)$ given by (4.3.14), with $z = \exp(-2K)$. In (a) the curve intercepts the straight line $y = x$ only once, at A. Two typical sequences of points $P_n = (x_n, y(x_n))$ are shown, one starting to the right of A, the other $\{P'_0, P'_1, P'_2, \ldots\}$ to the left. All such sequences converge to the limit point A. In (b) there are three intersections A, B, C. A sequence $\{P_n\}$ grows in the direction of the arrows, never crossing A, B or C. Thus A and C are stable limit points, B is an unstable fixed point.

We need some more convenient rule to determine which stable fixed point, A or C, is the one approached. The borderline case is when P_0 is the point B, i.e. when $x = 1$ is a solution of the equation $x = y(x)$. From (4.3.14) this occurs when, and only when, $h = 0$. If $h > 0$, then P_0 lies to the left of B so P_n tends to A. Conversely, if $h > 0$, then P_n tends to C.

Summarizing, when $n \rightarrow \infty$ the magnetization is given, using (4.3.11), by

$$M = \frac{e^{2h} - x^q}{e^{2h} + x^q}, \tag{4.4.1}$$

where x is a solution of

$$x = y(x). \tag{4.4.2}$$

If there are three solutions, the smallest must be chosen for $h > 0$, the largest for $h < 0$.

These equations can be written in a more conventional form by defining

$$z = \mathrm{e}^{-2K}, \quad \mu = \mathrm{e}^{-2h}, \quad \mu_1 = \mu x^{q-1}. \tag{4.4.3}$$

Then, using (4.3.14), (4.4.2) gives

$$x = (z + \mu_1)/(1 + \mu_1 z). \tag{4.4.4}$$

From (4.4.3), (4.4.4) and (4.4.1) it follows that

$$\mu_1/\mu = [(z + \mu_1)/(1 + \mu_1 z)]^{q-1}, \tag{4.4.5a}$$

$$M = (1 - \mu_1^2)/(1 + \mu_1^2 + 2\mu_1 z). \tag{4.4.5b}$$

The first of the equations (4.4.5) defines μ_1; the second gives the magnetization M. These are the same as the results of the Bethe approximation for a lattice of coordination number q (Domb, 1960, pp. 251–254).

4.5 Magnetization as a Function of H

Now suppose T, and hence K, is fixed and consider the variation of x and M with $h = H/kT$. Using (4.3.14) the equation (4.4.2) can be written

$$\mathrm{e}^{2h} = x^{q-1}(\mathrm{e}^{2K} - x)/(\mathrm{e}^{2K}x - 1). \tag{4.5.1}$$

All the x_n are positive, and so is the limit point x. For the RHS of (4.5.1) to be positive it follows that x must lie in the interval

$$\mathrm{e}^{-2K} < x < \mathrm{e}^{2K}. \tag{4.5.2}$$

Clearly (4.5.1) defines h as a function of x, for fixed K. (This function is of course not the same as the scaling function $h_s(x)$ of Section 1.2.) Differentiating (4.5.1) logarithmically gives

$$2x\frac{\mathrm{d}h}{\mathrm{d}x} = q - 1 - \frac{2\sinh 2K}{2\cosh 2K - x - x^{-1}}. \tag{4.5.3}$$

For x in the interval (4.5.2), the RHS of (4.5.3) has its maximum at $x = 1$. If this maximum is negative, i.e. if $K < K_c$, where

$$K_c = \tfrac{1}{2}\ln[q/(q-2)], \tag{4.5.4}$$

then h decreases monotonically from ∞ to 0 as x increases from $\exp(-2K)$ to $\exp(2K)$. Hence for given real h, (4.5.1) has one and only one real positive solution for x, and x is an analytic function of h for $-\infty < h < \infty$.

If, on the other hand, $K > K_c$, then dh/dx is positive for x sufficiently close to one. From (4.5.1), $h = 0$ when $x = 1$, so the function $h(x)$ has a graph of the type shown in Fig. 4.3.

For sufficiently small h, (4.5.1) therefore has three solutions for x. From the discussions of Section 4.4, if $h > 0$ the limit point of the sequence given by (4.3.1) corresponds to the smallest solution for x. If $h < 0$ it corresponds to the largest solution.

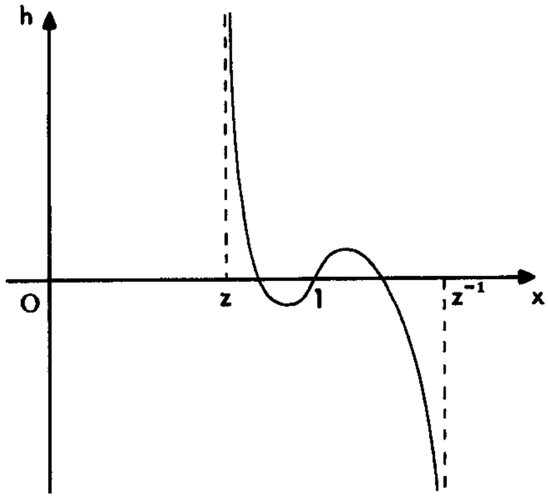

Fig. 4.3. A typical sketch of h as a function of x for $T < T_c$.

Considering the behaviour as h decreases from $+\infty$ through zero to $-\infty$, it is therefore apparent from Fig. 4.3 that x is an analytic function of h, except at $h = 0$, where it jumps discontinuously from the smallest to the largest solution.

In all cases x is a decreasing function of h, satisfying

$$x(-h) = 1/x(h) \,. \tag{4.5.5}$$

From (4.4.1) it follows that M is an odd function of h. It increases monotonically from -1 to 1 as h increases from $-\infty$ to ∞ and is analytic if $K < K_c$. If $K > K_c$, then it is analytic apart from a jump discontinuity at $h = 0$.

This is precisely the typical behaviour of a ferromagnet that was outlined in Section 1.1. Thus the Ising model on the Bethe lattice exhibits ferromagnetism, with a critical point at $H = 0$, $T = T_c$, where

$$J/kT_c = \tfrac{1}{2}\ln\,[q/(q-2)] \,. \tag{4.5.6}$$

4.6 Free Energy

The total free energy of the Cayley tree is

$$F = -kT \ln Z\,, \tag{4.6.1}$$

where Z is given by (4.3.1) and (4.3.2). Differentiating these equations with respect to $H = hkT$ gives

$$-\frac{\partial F}{\partial H} = \sum_i M_i\,, \tag{4.6.2}$$

where the summation is over all sites i and

$$M_i = \langle \sigma_i \rangle$$

is the local magnetization at site i. Each M_i is a function of H, and hence h, for given temperature T. To show this we shall sometimes write it as $M_i(h)$.

If H is large and positive the summation in (4.3.1) is dominated by the state with all spins up, so in this limit

$$F/kT = -KN_e - hN\,, \tag{4.6.3}$$

N_e being the number of edges and N the number of sites. Also, in this limit $\langle \sigma_i \rangle = 1$ for $i = 1, \ldots, N$.

We can now integrate (4.6.2) with respect to H, using (4.6.3) to obtain the integration constant. This gives

$$F/kT = -KN_e - hN + \sum_i \int_h^\infty [M_i(h') - 1]\, \mathrm{d}h'\,. \tag{4.6.4}$$

Alternatively, if q_i is the number of sites adjacent to site i, then $\sum_i q_i = 2N_e$, and (4.6.4) can be written

$$F = \sum_i f_i\,,$$

where

$$f_i/kT = -\tfrac{1}{2}Kq_i - h + \int_h^\infty [M_i(h') - 1]\, \mathrm{d}h'\,. \tag{4.6.5}$$

Each f_i can be thought of as the free energy of site i. For an homogeneous lattice the f_i are all equal to the usual free energy f, and on differentiating (4.6.5) one regains the usual relation (1.7.14).

As we remarked above, the difficulty with the Cayley tree is that it is not homogeneous, there being a significant number of boundary or near-boundary sites that have properties different from the interior. However, all sites deep inside the graph have the same local magnetization M, and hence the same local free energy f, given by (4.6.5). This free energy is therefore the free energy of the Ising model on the Bethe lattice. It is given by setting $q_i = q$, $M_i = M$ in (4.6.5), and using the equations (4.5.1), (4.4.1) for x and M as functions of h.

Noting that x is a monotonic differentiable function of h for $h > 0$, one can change the integration variable in (4.6.5) from h' to $x' = x(h')$. This gives [dropping the suffixes i and using $z = \exp(-2K)$]

$$f/kT = -\tfrac{1}{2}Kq - h - \int_z^x [M(x') - 1] \left[\frac{dh}{dx}\right]_{x=x'} dx' \qquad (4.6.6)$$

provided $h > 0$ (or $K < K_c$).

Substituting the expression (4.5.1) for $\exp(2h)$ into (4.4.1), and using (4.5.3), the integrand in (4.6.6) can be written, after a little re-arrangement, as

$$\frac{z}{1 - zx'} - (q-2)\frac{x' - z}{1 + x'^2 - 2x'z}. \qquad (4.6.7)$$

This can be easily integrated to give, eliminating h by using (4.5.1),

$$f/kT = -\tfrac{1}{2}Kq - \tfrac{1}{2}q \ln(1 - z^2)$$
$$+ \tfrac{1}{2}\ln[z^2 + 1 - z(x + x^{-1})] + \tfrac{1}{2}(q-2)\ln(x + x^{-1} - 2z). \qquad (4.6.8)$$

Negating h has the effect of inverting x, which leaves (4.6.8) unchanged. Since f must be an even function of h, it follows that (4.6.8) is true for all real h. Together with the equation (4.5.1) for x, it gives the free energy per site of the Ising model on the Bethe lattice.

4.7 Low-Temperature Zero-Field Results

A problem arises with *any* ferromagnetic Ising model if $H = 0$ and $T < T_c$. In this case the spins do not know whether to be mostly up, or mostly down. If just the boundary spins are fixed to be up, every spin will have a greater probability of being up than down. In a sense the 'thermodynamic limit' does not exist, since the bulk properties depend on the boundary conditions.

This is particularly evident in the present model: if $H = 0$ then it is obvious from (4.3.13)–(4.3.15) that $x_n = 1$, for all n. If $T < T_c$ this means that all the points $P_n = (x_n, y_n)$ are the point B in Fig. 4.2(b). However, this is an unstable fixed point of (4.3.13): if x_0 is not one, but just less than one, then the sequence $\{P_n\}$ will converge not to B, but to the stable limit point A.

There are at least two ways round this difficulty: one is to take $H = 0$ and fix all boundary spins up; the other to take $H > 0$, let $n \to \infty$, and *then* let $H \to 0_+$. In either case the sequence $\{P_n\}$ will converge to A and the limiting value of x is, from (4.4.2) and (4.3.14), the smallest positive solution of the equation

$$z = \mathrm{e}^{-2J/kT} = x \frac{1 - x^{q-2}}{1 + x^{q-2}}. \tag{4.7.1}$$

If $T < T_c$, this value of x is less than one. From (4.4.1) and (4.6.8) the spontaneous magnetization M and free energy f are then given by

$$M = \frac{1 - x^q}{1 + x^q}, \tag{4.7.2}$$

$$\mathrm{e}^{-f/kT} = (1 + x^q)\left[\frac{(1 - x^q)(1 - x^{2q-2})^2}{x(1 - x^{q-2})(1 - x^{2q})^2}\right]^{q/4}. \tag{4.7.3}$$

It is interesting to compare these results with those of the two-dimensional Ising model. This will be done in Section 11.8.

4.8 Critical Behaviour

Set $x = \exp(-2s)$, then (4.5.1) becomes

$$h = -(q-1)s + \tfrac{1}{2}\ln[\sinh(K+s)/\sinh(K-s)], \tag{4.8.1}$$

which makes it clear that h is an odd function of s. Taylor expanding, we obtain

$$h = [\coth K - q + 1]s + \tfrac{1}{3}\coth K \operatorname{cosech}^2 K \, s^3 + \ldots. \tag{4.8.2}$$

The critical value of K is given by (4.5.4), i.e. by $\coth K_c = q - 1$. Setting as usual

$$t = (T - T_c)/T_c, \tag{4.8.3}$$

and using $K = J/kT$, it follows that for t small

$$\coth K - q + 1 = q(q-2)K_c t + \mathcal{O}(t^2). \tag{4.8.4}$$

Using this result in (4.8.2), together with $h = H/kT$, gives (for t and s small):

$$H/kT_c = q(q-2)\{K_c ts + \tfrac{1}{3}(q-1)s^3 + \mathcal{O}(t^2 s, ts^3, s^5)\}. \tag{4.8.5}$$

From (4.4.1), the magnetization M is given by

$$M = \tanh(h + qs). \tag{4.8.6}$$

From (4.8.5), h is much less than s, which is itself small, so $M \simeq qs$, or conversely

$$s = q^{-1}M + \mathcal{O}(h, M^3). \tag{4.8.7}$$

Substituting this result into (4.8.5) and neglecting terms of order t^2M, tM^3 or M^5, we obtain

$$H/kT_c = M^3 h_s(t/M^2), \tag{4.8.8}$$

where

$$h_s(x) = \tfrac{1}{2}(q-2)x\ln[q/(q-2)] + (q-1)(q-2)/(3q^2). \tag{4.8.9}$$

Comparing (1.2.1) and (4.8.8), we see that the scaling hypothesis is satisfied for this model, $h_s(x)$ being the scaling function. It is linear, and critical exponents β and δ have the values

$$\beta = \tfrac{1}{2}, \quad \delta = 3. \tag{4.8.10}$$

Thus all the exponents β, δ, α, α', γ, γ' must have the same values as those of the mean-field model (Section 3.3), i.e. the 'classical' values.

All the above results are very similar to those of the mean-field model of Chapter 3. (In fact they are the same in the limit $q \to \infty$, qK finite.) However, the Bethe-lattice model is really much more respectable than the mean-field one: its interactions are independent of the size of the system, and each spin interacts only with its nearest neighbours.

4.9 Anisotropic Model

The key equations (4.3.14), (4.4.2), (4.4.1), (4.6.8) of the above working can be summarized (using the first two to eliminate z from the last) as

$$z = \exp(-2K) = (x - \mu x^{q-1})/(1 - \mu x^q), \tag{4.9.1}$$

$$M = (1 - \mu x^q)/(1 + \mu x^q), \tag{4.9.2}$$

$$-f/kT = h + \tfrac{1}{2}qK + \ln(1 + \mu x^q)$$
$$+ \tfrac{1}{2}q \ln[(1 - \mu^2 x^{2q-2})/(1 - \mu^2 x^{2q})] \,. \qquad (4.9.3)$$

The edges of the Bethe lattice can be grouped into classes $1, \ldots, q$, so that each site lies on just one edge of each class. Then the interaction coefficient K can be given a different value for different classes of edges. If K_r is its value for class r (where $r = 1, \ldots, q$), then this anisotropic model can also be solved by the above methods.

The equations (4.9.1)–(4.9.3) generalize to

$$z_r = \exp(-2K_r) = (x_r - tx_r^{-1})/(1 - t) \,, \quad r = 1, \ldots, q \,, \qquad (4.9.4a)$$

$$\mu = \exp(-2h) = t/(x_1 \ldots x_q) \,, \qquad (4.9.4b)$$

$$M = (1 - t)/(1 + t) \qquad (4.9.5)$$

$$-f/kT = h + \tfrac{1}{2}(K_1 + \ldots + K_q) + \ln(1 + t) + \tfrac{1}{2} \sum_{r=1}^{q} \ln \frac{1 - t^2 x_r^{-2}}{1 - t^2} \,. \qquad (4.9.6)$$

These define M, f as functions of $K_1, \ldots, K_q, h$; the parameters $x_1, \ldots, x_q$, t being defined by (4.9.4). The critical point occurs when $h = 0$ and $x_1, \ldots, x_q$, t are infinitesimally different from one. From (4.9.4) this implies that

$$\exp(-2K_1) + \ldots + \exp(-2K_q) = q - 2 \,. \qquad (4.9.7)$$

[This result is derived in (11.8.37)–(11.8.42).]